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COMBINED LAMINAR FREE AND FORCED CONVECTION HEAT TRANSFER TO NON-NEWTONIAN FLUIDS

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Abstract—A theoretical analysis for the combined laminar free and forced convection heat transfer to non-Newtonian fluids in external flows is presented. The parameter controlling the free and forced convection is found to be $N_{Gr'}/N_{Re'}^{2/2-n}$, where n is the flow behaviour index. Numerical solution of the transformed boundary-layer equations has been carried out for the case of a flat plate with uniform wall temperature, for different flow behaviour indices and different values of the controlling parameter. The results of heat transfer, shear stress, velocity and temperature profiles as functions of flow behaviour index and the controlling parameter at different Prandtl numbers are discussed for opposing and aiding flows.

NOMENCLATURE

A, constant;
B, constant;
C, $[2^{n+1}K/A^{2-n}\rho]$;
C_p, specific heat [Btu/lb °F];
g, gravitational constant [ft/h²];
k, thermal conductivity [Btu/h ft °F];
K, n, parameters in power law model as defined in equation (1);
m, constant;
n', constant;
$N_{Gr'}$, modified Grashof number;
N_{Nu}, Nusselt number;
$N_{Pr'}$, modified Prandtl number, $=(C_p U_\infty \rho x)/k(N_{Re'}^{2/(1+n)})$;
$N_{Re'}$, modified Reynolds number, $= (\rho U_\infty^{2-n} x^n/K)$;
T, temperature [°K];
u, component of velocity in direction of x-axis [ft/s];
v, component of velocity in direction of y-axis [ft/s];
x, distance in system of Cartesian coordinates [ft];
y, distance in system of Cartesian coordinates [ft].

Greek symbols

β, coefficient of volumetric expansion [$°K^{-1}$];
η, dimensionless coordinate;
θ, dimensionless temperature ratio;
ρ, density [lb/ft³];
τ_{xy}, shear stress [lb/ft²];
τ_w, shear stress at wall [lb/ft²].

INTRODUCTION

THE INCREASING use of many non-Newtonian fluids in chemical process industries has necessitated to understand their behaviour in many transport processes. The prediction of heat-transfer rates and shear stress for external flows of non-Newtonian fluids, is therefore, of practical interest.

While applying the numerical and analytical tools to solve the problems of fluid-dynamics and heat transfer for non-Newtonian fluids, a number of complications arise due to the non-linear relation between the stress tensor and deformation strain tensor. There are various empirical models suggested in the literature for non-Newtonian fluids. If one ignores the effect of normal stresses in the generalized constitutive equation [1], for a two dimensional flow, it reduces to a power-law model. The power-law model of Ostwald-de-Waele characterized by the equation

$$\tau_{xy} = -K\left(\frac{\partial u}{\partial y}\right)^n \tag{1}$$

where n is the flow behaviour index and K is the consistency index, is the one which is most commonly used and is fully discussed by Reiner [2], Fredrickson [3] and Metzner [4].

Most research has been concentrated on the flow of non-Newtonian fluids in circular and non-circular pipes, and channels. The simpler classical problems of fluid dynamics and heat transfer in pipes have been critically reviewed by Metzner [5], and will not be discussed in this paper.

Acrivos *et al.* [6] have presented the theoretical analysis of flow past an external surface of power-law fluids under forced convection. They applied similarity variables to the momentum equations. Local heat-transfer rates were estimated, using Lighthill's approximate formula, on the assumption that $N_{Pr'} \to \infty$, i.e. the thermal boundary layer is much thinner than the shear layer. Although Prandtl numbers for non-Newtonian fluids are higher than one, the possibility of solving such equations without restriction will likely give more insight into the problem especially when different mechanisms of heat transfer like forced and free convection are coupled.

By reducing three dimensional boundary-layer equations to ordinary differential equations, Schowalter [7] has shown that similar solutions exist for externals flows (of power-law non-Newtonian fluids) over a wedge.

Ames and Lee [8] have critically reviewed the existing important similar solutions available in literature, for boundary-layer flows of non-Newtonian fluids. They have applied transformation group method to obtain similarity variables and equations for various types of flows, e.g. Falkner–Skan flows and Goldstein flows. The numerical solutions of the forced convection of power-law fluids for a right-angle wedge with an isothermal surface have been presented. They solved the non-linear differential equations by using continuous analytic continuation method, with a built-in iterative procedure.

Acrivos [9] analyzed the problem of natural convection heat transfer to power-law fluids for different geometrical configurations. For a large Prandtl number ($N_{Pr'} > 10$) he proposed that local heat-transfer coefficient can be calculated from

$$N_{Nu} = C.\ N_{Gr'}^{\frac{1}{2(n+1)}}\, N_{Pr'}^{\frac{n}{(3n+1)}} \tag{2}$$

where $C = 0{\cdot}55$. This correlation was later varified by Tien *et al.* [15] using 0·5 and 1 per cent carbopol solutions.

Na and Hansen [10] have reported that the similarity solutions of the laminar natural convection flow of non-Newtonian fluids are possible by applying group theoretic methods. Hence, the limitations on the power-law model representing the boundary-layer flow under natural convection conditions pointed out by Acrivos [9] have been removed by applying two distinct classes of transformations, namely a linear group of transformations and a spiral group.

Berkowski [11] has considered the transformation of the boundary-layer equations in general for pseudoplastic and dilatant fluids. However, rigorous solutions and the implications of various dimensionless groups encountered for different problems have not been analyzed. Finally, Acrivos [12] has critically reviewed the state of knowledge for the combined effect of forced and free convection heat transfer for Newtonian fluid under laminar boundary-layer flows. He has established that there exist two different controlling parameters;

$$\text{(i)}\quad \frac{N_{Gr}}{N_{Re}^2\, N_{Pr}^{\frac{1}{3}}} \quad \text{for} \quad N_{Pr} \to \infty,$$

(ii) $\frac{N_{Gr}}{N_{Re}^2}$ for $N_{Pr} \to 0$.

Results of shear stress and heat transfer have been presented for stagnation flow under two asymptotic conditions.

In the present work, an analysis of laminar boundary-layer equations for non-Newtonian fluids under the influence of both forced and free convection is made. A numerical technique has been developed in solving these coupled non-linear differential equations. The interactions of the two convective mechanisms will then be analyzed in detail for the case of a flat plate at different Prandtl numbers. Furthermore, their combined effects on the drag coefficients and heat-transfer rates will be discussed.

For two-dimensional flows

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_\infty \frac{dU_\infty}{dx} + g_x\beta(T - T_\infty) + \frac{K}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^n \quad \text{equation of motion} \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{equation of continuity} \tag{4}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{C_p\rho}\frac{\partial^2 T}{\partial y^2} \quad \text{equation of energy} \tag{5}$$

with the boundary conditions

$$y = 0, \qquad u = 0, \qquad T = T_w$$

$$y \to \infty, \qquad u \to U_\infty, \qquad T \to T_\infty.$$

Let the free stream velocity and temperatures be connected by

$$U_\infty = Ax^m; \qquad T_w - T_\infty = Bx^{n'} \tag{6}$$

where U_∞ and T_∞ are free stream velocity and temperature respectively, T_w is the surface temperature, and A, B and n' are constants. The following dimensionless groups are then introduced

$$N_{Re'} = \frac{\rho U_\infty^{2-n} x^n}{K}$$

$$N_{Gr'} = \frac{\rho^{2/(2-n)} x^{n+2/(2-n)} [\beta g(T_w - T_\infty)]}{K^{2/2-n}}$$

$$N_{Pr'} = \frac{C_p U_\infty \rho x}{k(N_{Re'})^{2/(1+n)}}$$

$$\eta = \frac{y}{2x}\left[\frac{U_\infty^{2-n}\rho x^n}{K}\right]^{1/(n+1)}$$

$$u = \frac{U_\infty}{2} f'(\eta)$$

$$v = CAx^{2mn-m+1/n+1}x^{-1}\left[\left\{\frac{m(n-2)+1}{n+1}\right\} \times \eta f' - \left(\frac{2mn-m+1}{n+1}\right) f\right].$$

After the transformations, the dimensionless form of boundary-layer equations, hence, will be

$$nf'''(f'')^{n-1} + 4m(2^{2n-1}) \pm 4(2^{2n-1})\frac{N_{Gr'}}{N_{Re'}^{2/2-n}}\,(\theta) - m(2^{2n-1})(f'^2) + ff''\left(\frac{2mn-m+1}{n+1}\right) = 0 \tag{7}$$

and

$$\theta'' + N_{Pr'}\left[2f\theta'\frac{2mn-m+1}{n+1} - 2n'f'\theta\right] = 0 \tag{8}$$

with boundary conditions:

$$\eta = 0, \quad f = 0, \quad f' = 0, \quad \theta = 1,$$

$$\eta \to \infty, \quad f' \to 2, \quad \theta \to 0.$$

For Newtonian fluids, these equations reduce to the equations proposed by Sparrow *et al.* [13] except the restriction $n' = 2m - 1$ in their paper has been removed.

It can be found by inspection that for $0 < n < 2$, the increase in U_∞ corresponds to an increase in the value of Reynolds number. With $n = 0$ and $n = 2$, the value of Reynolds number does not depend on n and U_∞ respectively, while $n > 2$, increase of U_∞ corresponds to a decrease of Reynolds number. The controlling parameter, P, is defined as $N_{Gr'}/N_{Re'}^{2/2-n}$, corresponding to $N_{Gr'}/N_{Re}^2$ for Newtonian fluids. This parameter P characterizes the effect of forced and free convection in non-Newtonian boundary-layer flows. P decreases in the range $0 < n < 2$, since Reynolds number is increased, with an increase of U_∞. Therefore under sufficiently lower values of P, the effect of forced convection is predominant whereas at higher values of P, in the range $0 < n < 2$, the effect of free convection is predominant. These equations (7) and (8) become uncoupled when $n = 2$, since the value of P will be negligible. When $n > 2$, the boundary-layer flows are not of practical interest and do not represent the asymptotic state of laminar motion, which is obtained as U_∞ is made sufficiently large. The parameter P will be positive for aiding flows and negative for opposing flows.

For a given geometry the important parameters in the system of equations (7) and (8) are two dimensionless groups $N_{Gr'}/N_{Re'}^{2/(2-n)}$ and $N_{Pr'}$. The expressions for shear stress and Nusselt number can be written as

$$\frac{\tau_w}{(\rho U_\infty^2/2)} N_{Re'}^{1/(n+1)} = f_1 \left[N_{Pr'}, \frac{N_{Gr'}}{N_{Re'}^{2/(2-n)}} \right] \tag{9}$$

$$\frac{N_{Nu}}{N_{Re'}^{1/(n+1)}} = f_2 \left[N_{Pr'}, \frac{N_{Gr'}}{N_{Re'}^{2/(2-n)}} \right]. \tag{10}$$

The functions f_1 and f_2 are determined only by solving the equations (7) and (8). When the parameter P approaches zero or infinity the equations reduce to corresponding forced and free convection, respectively. The form of expression for f_1 and f_2 under two limiting conditions $P \to 0$ and $P \to \infty$, for two asymptotic conditions $N_{Pr'} \to 0$ and $N_{Pr'} \to \infty$, can be obtained by solving (7) and (8) for various P and Prandtl numbers.

For lower values of Prandtl number the equations (7) and (8) are solved very easily by using the conventional Runge–Kutta method with extended stability described by Lawson [14]. A built-in iterative scheme facilities to make the solution to be convergent in fewer number of iterations and also to arrive at the correct initial values of f'' and θ'. In all the cases the convergence can be noted by inspecting that $\theta \to 0$, $f' \to 2$, $f'' \to 0$ as $\eta \to \infty$. The equation (7) can be written as (for a flat plate with constant wall temperature)

$$\theta = 1 - \frac{\int_0^\eta \exp\left\{-2N_{Pr'} \int_0^\eta \frac{f}{n+1}\,d\eta\right\} d\eta}{\int_0^\infty \exp\left\{-2N_{Pr'} \int_0^\eta \frac{f}{n+1}\,d\eta\right\} d\eta}. \tag{11}$$

As we see from equation (11), the values of $\theta'(0)$ depend on $f''(0)$ and $N_{Pr'}$. For lower values of $N_{Pr'}$, one can choose a large value of η and readily get a convergent solution. However, at higher values of $N_{Pr'}$, although the convergent values can be obtained using the same step size as for a smaller $N_{Pr'}$, the convergent values were not stable. Hence, smaller step size even as much as thirty-two times less than the step size used for smaller $N_{Pr'}$ was used, in order to get stable convergent values at higher $N_{Pr'}$. They agree very well with those reported for Newtonian fluid boundary-layer solutions under forced convection [16].

Solutions have been carried out for the values of controlling parameter ranging from 0·1 to 5 and $N_{Pr'}$ ranging from 0·7 to 1000, for a flat plate with uniform wall temperature for aiding flows. For opposing flows P varies from 0·1 to 0·3. The numerical values of $f''(0)$ and $\theta'(0)$ are tabulated for different flow behaviour index and different P at different Prandtl numbers.

The local rate of heat transfer is calculated using

$$q = -k\left(\frac{\partial T}{\partial y}\right)y = 0 \quad (12)$$

i.e.

$$\frac{N_{Nu}}{N_{Re'}^{1/(n+1)}} = -\tfrac{1}{2}\theta'(0) \quad (13)$$

and shear stress is calculated using

$$C_f = \tau_w \frac{(\rho U_\infty^2)}{2} \quad (14)$$

i.e.

$$N_{Re'}^{1/(n+1)} = C_f = 1/2^{2n-1}[f''(0)]^n. \quad (15)$$

These expressions are the same as those of Sparrow *et al.* [12] for Newtonian fluids.

RESULTS AND DISCUSSION

The solutions of equations (7) and (8) are included in Table 1. The numbers appearing there have been rounded and they do not represent the eight figures actually used in order to satisfy the boundary conditions for large η.

The values of $f, f', f'', \theta, \theta'$, as a function of n, are available upon request.

In Figs. 1–4, some typical velocity and temperature profiles with different values of n, and P, at various Prandtl numbers are plotted. When $n = 1$, the velocity and temperature profiles correspond to Newtonian fluid. It can be observed from these figures that the flow behaviour index has a very significant effect. The thermal boundary layer will be thinner for a pseudoplastic fluid than the corresponding thermal boundary layer for Newtonian and dilatant fluids. This observation is true at lower values of Prandtl numbers and when forced convection effect is predominant. However, when free convection effect is important and Prandtl number is high, reverse behaviour is observed. Hence, the behaviour of non-Newtonian boundary layer under mixed flow conditions will be different from the corresponding pure flow conditions, at different Prandtl numbers.

The velocity profile for a Newtonian fluid will be much steeper than the corresponding velocity profile for pseudoplastic and dilatant fluids at low Prandtl numbers and when the effect of forced convection is predominant. But under the same conditions, at higher Prandtl numbers, the velocity profile for a pseudoplastic is much steeper. Similar conclusion holds when the effect of free convection is important.

In Figs. 5–8 the results of shear stress and heat transfer are plotted for different flow behaviour index n and the controlling parameter P, at different Prandtl numbers, for aiding flows. It is apparent from these figures that the drag coefficient is higher for pseudoplastic fluid than that of dilatant fluid, when the effect of forced convection is important (i.e. at lower values of P). At higher values of P, when natural convection is dominating, the reverse is true.

The effect of Prandtl numbers and flow behaviour index on the shear stress can be observed from Figs. 9 and 10. It can be observed that as Prandtl numbers increase, the drag coefficients decrease but they increase with the flow behaviour index, attaining a maximum value when $n = 1{\cdot}5$. The local heat-transfer rates are higher for pseudoplastic fluid than those of Newtonian and dilatant fluids.

From Figs. 5–8 it can be observed that the local rates of heat transfer are increased at higher Prandtl numbers and influenced by non-Newtonian behaviour. The theoretical analysis of the effect of natural convection on non-Newtonian laminar boundary layer has been presented by Acrivos [9] as

$$N_{Nu} = C[N_{Gr'}^{(2-n)/[2(n+1)]}][N_{Pr'}^{n/(3n+1)}] \quad \text{for } N_{Pr'} > 10 \quad (16)$$

where $C = 0{\cdot}55$.

When $n = 1$ this equation reduces to the correlation [14] of free convection

$$N_{Nu} = C[N_{Gr}^{\frac{1}{4}}][N_{Pr}^{\frac{1}{4}}] \quad (17)$$

for Newtonian fluids. Furthermore, recent

Table 1. Results of shear stress and heat transfer

n	P	$f''(0)$	C_f	$-\theta'(0)$
		$N_{Pr'} = 1·00$		
0·2	5·0	131·9788	4·024	1·9508
	1·0	16·5759	2·658	1·2641
	0·1	3·0833	1·889	0·9471
	−0·1	1·0239	1·523	0·7949
	−0·175	0·5025	1·318	0·7233
	−0·25	0·1437	1·027	0·6273
	−0·3	0·0235	0·7115	0·5435
0·4	5·0	43·9316	5·216	1·5894
	1·0	8·5887	2·715	1·0866
	0·1778	2·5766	1·676	0·8496
	0·1	2·1267	1·557	0·8242
0·6	5·0	26·7094	6·248	1·4362
	1·0	6·4899	2·674	0·9933
	0·1	1·8759	1·270	0·7632
0·8	5·0	19·4630	6·931	1·3161
	1·0	5·6867	2·650	0·9415
	0·45	3·4960	1·797	0·8378
	0·1	1·8107	1·060	0·7270
	0·045	1·5518	0·7449	0·7069
1·0	5·0	15·6	7·8	1·22
	0·5	3·5371	1·7685	0·8177
	0·1	1·8458	0·9299	0·7072
	−0·10	0·7105	0·3553	0·6021
	−0·13	0·4831	0·2416	0·5749
	−0·15	0·3069	0·1535	0·5513
	−0·175	0·1764	0·0882	0·5059
1·5	5·0	11·0447	9·173	1·0702
	0·5	3·4189	1·650	0·7705
	0·1	2·0236	0·7194	0·6484
	−0·1	0·5775	0·1097	0·5600
	−0·11	0·4126	0·0662	0·5456
	−0·12	0·1545	0·0152	0·5354
0·2	5·0	6·3644	2·195	2·6735
	0·1	2·2595	1·784	2·0387
0·4	5·0	6·0340	2·356	2·4712
	0·1	1·8775	1·478	1·8177
0·6	5·0	6·5733	2·695	2·3997
	0·1	1·7204	1·205	1·6838
1·0	5·0	9·9692	4·9846	2·4551
	0·1	1·6087	0·8043	1·5148
1·5	5·0	28·1498	37·33	2·8264
	0·1	1·6509	0·5240	1·3900
		$N_{Pr'} = 100$		
0·2	5·0	4·7231	2·067	5·3075
	0·1	2·1658	1·769	4·3631
0·4	5·0	4·3328	2·065	4·8576
	0·1	1·7861	1·445	3·8830

Table 1—continued

n	P	$f''(0)$	C_f	$-\theta'(0)$
		$N_{Pr'} = 10·0$		
0·6	5·0	4·5353	2·157	4·6648
	0·1	1·6188	1·161	3·5888
1·0	5·0	6·1917	3·0958	4·6506
	0·1	1·4671	0·7335	3·2095
1·5	5·0	14·8332	14·29	5·3206
	0·1	1·3820	0·4055	2·8932
		$N_{Pr'} = 1000$		
0·2	5·0	3·6090	1·956	10·6286
	0·1	2·1178	1·800	9·3583
0·4	5·0	3·2058	1·830	9·6366
	0·1	1·7390	1·432	8·3204
0·6	5·0	3·2211	1·756	9·1437
	0·1	1·5666	1·140	7·6805
1·0	5·0	3·9270	1·9634	8·8242
	0·1	1·3962	0·6980	6·8462
1·5	5·0	7·4083	5·041	9·5136
	0·1	1·3471	0·3909	6·2516

calculations of Schuh [17] and Pohlhausen [16] indicate the dependence of C on N_{Pr}. In order to compare the results of present investigations with those of Schuh [17] and Pohlhausen [16], Table 2 has been prepared.

From Table 2 one can observe that the trend of the results reported by Schuh [17] namely that C depends on $N_{Pr'}$ for pure natural convection, is also true for mixed flow conditions. But the values of C at $P = 5·0$ are lower than the values reported by Schuh [17]. This difference indicates that the effect of forced convection is still important at $P = 5·0$. Therefore in mixed flow, the value of C depends on the parameter P and $N_{Pr'}$, a point which has been very well established for Newtonian fluids by Acrivos [9], who claims that different parameters namely P and $P/N_{Pr'}^{1/3}$ control the flow regime as $N_{Pr'} \rightarrow 0$ and $N_{Pr'} \rightarrow \infty$, respectively. For non-Newtonian fluids the theoretical analysis of Acrivos [8] on the effect of natural convection on laminar

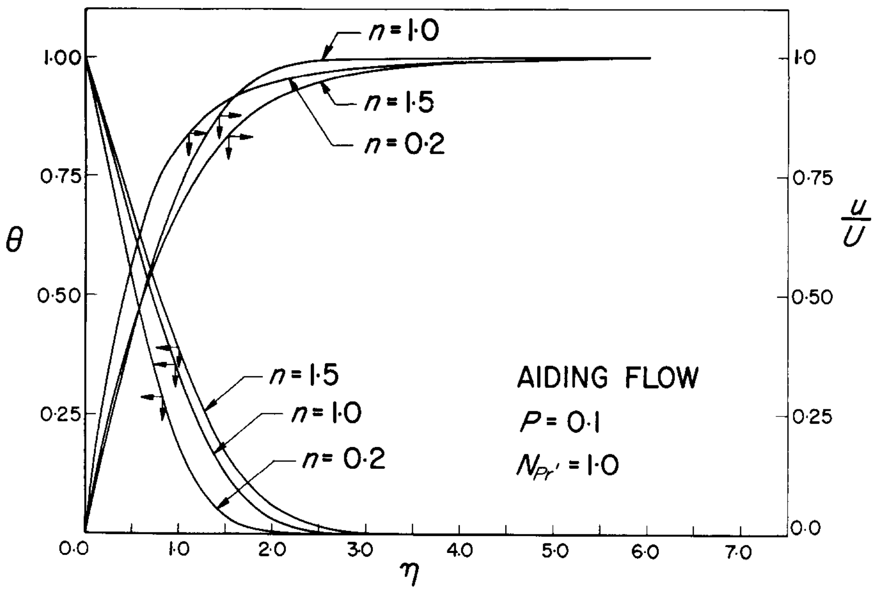

FIG. 1. Representative velocity and temperature profiles.

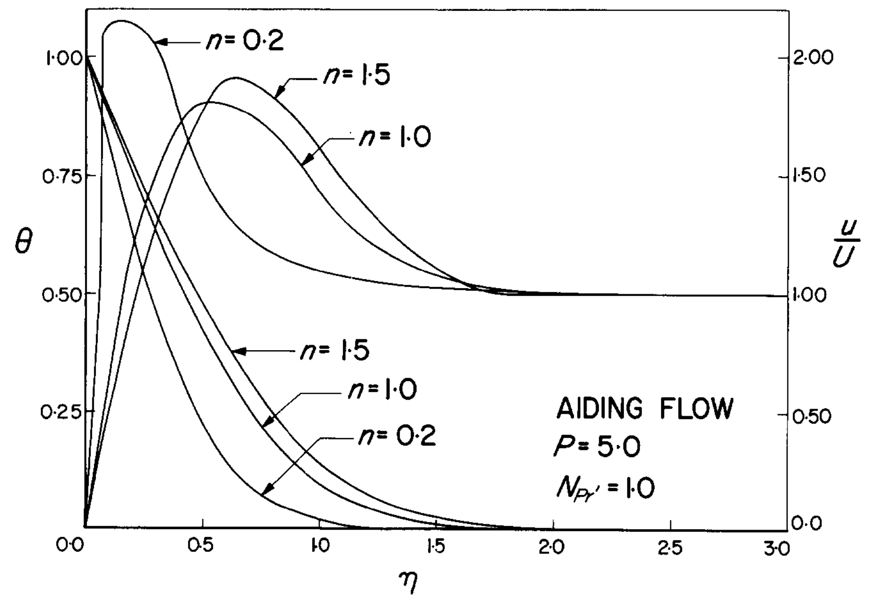

FIG. 2. Representative velocity and temperature profiles.

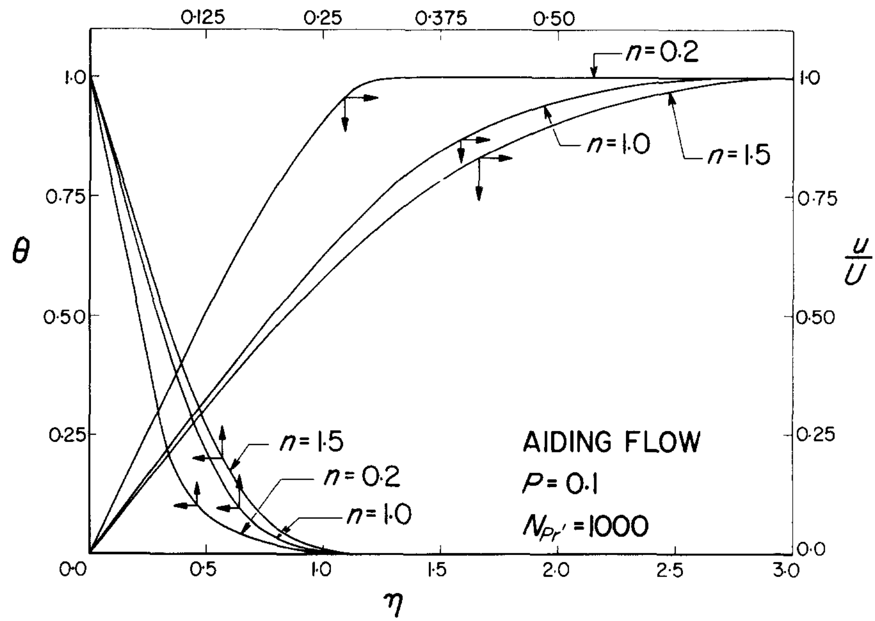

FIG. 3. Representative velocity and temperature profiles.

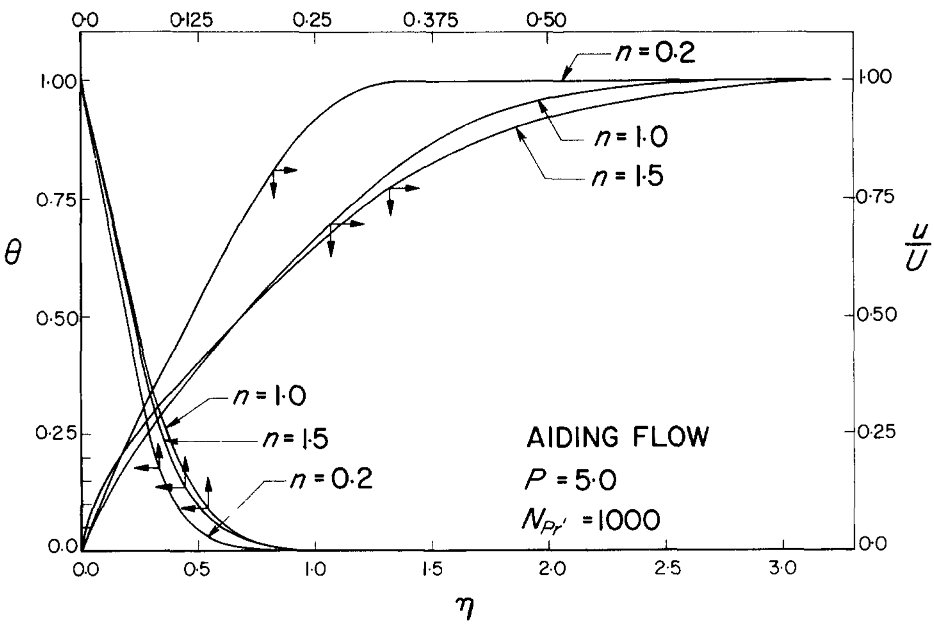

FIG. 4. Representative velocity and temperature profiles.

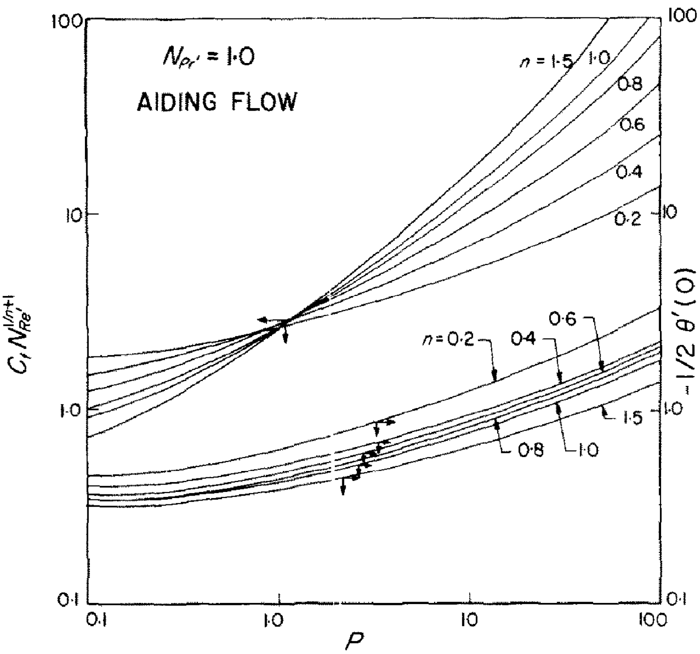

FIG. 5. Plot of shear stress and heat-transfer results.

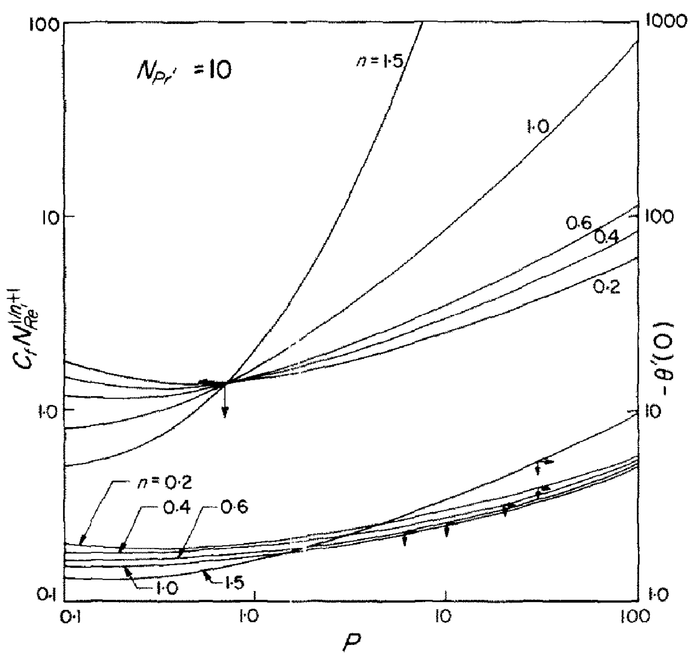

FIG. 6. Plot of shear stress and heat-transfer results.

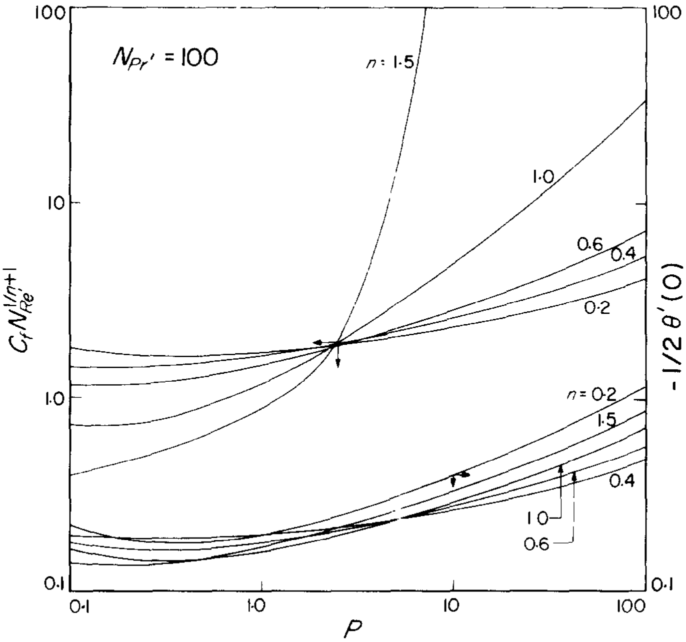

FIG. 7. Plot of shear stress and heat-transfer results.

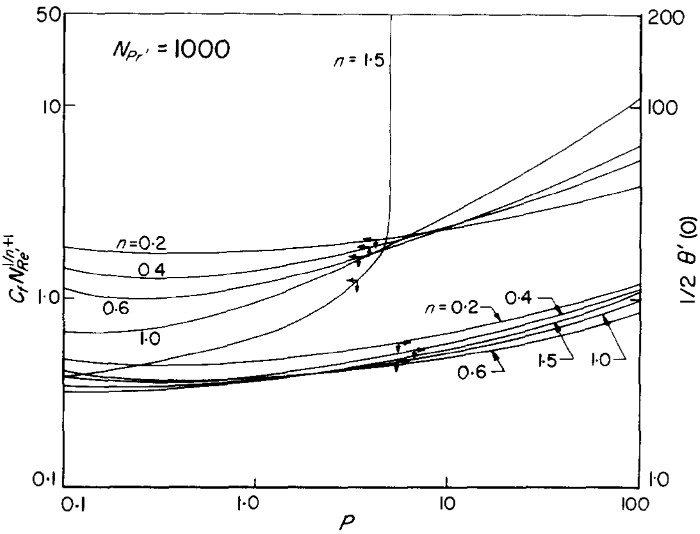

FIG. 8. Plot of shear stress and heat-transfer results.

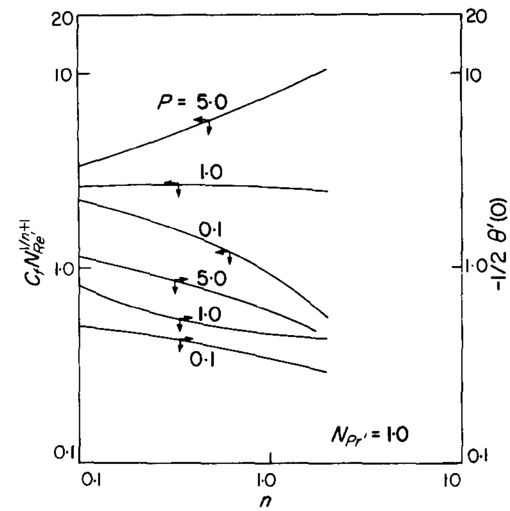

FIG. 9. Plot of shear stress and heat-transfer results vs. flow behaviour index.

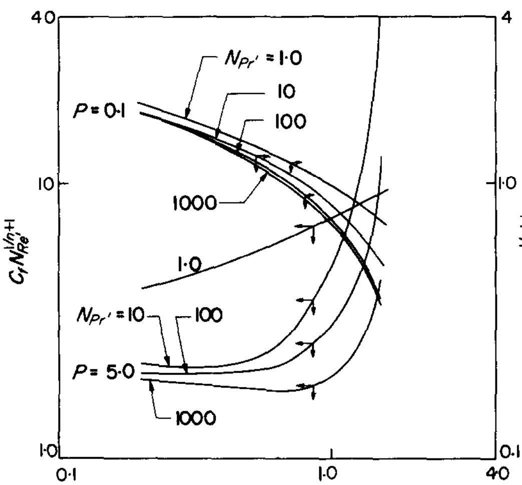

FIG. 10. Plot of shear stress results vs. flow behaviour index to show the effect of $N_{Pr'}$.

boundary layer and also the experimental work of Tien *et al.* [15], have not considered the dependence of C on $N_{Pr'}$. But in the present analysis it has been noted that at $P = 5{\cdot}0$, $n = 0{\cdot}6$, the value of C depends on $N_{Pr'}$ also. Hence one can conclude that for pure natural convection, the value of C depends on $N_{Pr'}$ for non-Newtonian fluids.

In the present analysis, the behaviour of boundary layer at higher values of P, will approximate to the boundary-layer behaviour under the influence of pure natural convection. With this view in mind, Table 3 has been prepared for comparison.

From Table 3 one can observe that the local rates of heat transfer for mixed flow at $P = 5{\cdot}0$ are correspondingly lower than the asymptotic values for natural convection, indicating that the effect of forced convection is still important. Higher values of controlling parameter P are needed to reach the asymptotic heat-transfer rates of natural convection. This observation agrees well with the result of Acrivos and others, critically reviewed by Acrivos [12], that forced convection will have negligible influence only when $P > 100$.

For non-Newtonian fluids, the similar observations are true and heat-transfer rates are higher for pseudoplastic than for Newtonian fluid. The difference in the local rates of heat transfer in mixed and pure flows appears to diminish for Newtonian and non-Newtonian fluids at higher Prandtl numbers, indicating that the pure flow criterion will be applicable at relatively lower values of P, for higher Prandtl numbers.

With a view to analyzing the importance of forced convection effects, in mixed flows, Table 4 has been prepared.

It can be found from Table 4 that at very low values of P, the local heat-transfer rates agree quite well with the forced convection asymptotic values, for Newtonian fluids [16], indicating that the effect of forced convection is dominant at $P = 0{\cdot}1$.

For non-Newtonian fluids, the local heat-transfer rates are higher than the corresponding values for Newtonian fluids. The calculated values from the asymptotic formula proposed by Acrivos *et al.* [6] are also included for comparison. They are much higher than the corresponding asymptotic values for Newtonian fluids and the values obtained in the present investigation. This deviation throws considerable

Table 2. Values of C from equations (16) *and* (17)

$N_{Pr'}$	Newtonian fluid		Non-Newtonian fluid	
	Schuh [17] and Pohlhausen [16]	present results $P - 5{\cdot}0$	present results $P = 5{\cdot}0, n = 0{\cdot}6$	Acrivos [8] $P = \infty, n = 0{\cdot}6$
1000	0·653	0·5243	0·5144	0·55
100	0·652	0·4916	0·4294	0·55
10	0·612	0·4616	0·4620	0·55
1·0		0·4072		
0·7	0·517	0·4047		

Table 3. $N_{Nu}/N_{Re}^{1/(n+1)} = \frac{1}{2}\theta'(0)$

$N_{Pr'}$	Newtonian fluid		Non-Newtonian fluid	
	present results $P = 5{\cdot}0$	asymptotic value [16] $P = \infty$	present results $P = 5{\cdot}0$	asymptotic value [8] $P = \infty$
1000	4·4441	4·626	4·5718	4·887
100	2·325	2·601	2·33	2·984
10	1·2275	1·463	1·20	1·309
1·0	0·61	0·8223	0·72	0·111

Table 4. $\frac{N_{Nu}}{N_{Re}^{1/(n+1)}} = -\frac{1}{2}\theta'(0)$

$N_{Pr'}$	Newtonian fluid $n = 1$		Non-Newtonian fluid $n = 0{\cdot}6$	
	$P = 0{\cdot}1$	asymptotic value [17] $P = 0$	$P = 0{\cdot}1$	asymptotic value [6] $P = 0$
1000	3·423	3·39	3·84	8·92
100	1·60	1·573	1·794	2·8225
10	0·75	0·75	0·84	0·89
1	0·35	0·34		

doubt on the validity of the assumptions made by Acrivos *et al.* [6].

Acrivos [10] has also proposed an approximate formula to calculate the local Nusselt number for mixed flow at higher Prandtl numbers, for Newtonian fluid, as

$$N_{Nu} \cong (N_{Nu0}^4 + N_{Nu\infty}^4)^{\frac{1}{4}} \quad (18)$$

where 0 and ∞ refer to forced and free convection asymptotes. Table 5 has been prepared to compare the values of present investigation with approximate formula, which does not account for the parameters P and N_{Pr}.

Table 5. $-\frac{1}{2}\theta'(0)$

$N_{Pr'}$	Newtonian fluid				
	pure forced convection [17]	pure free convection [16]	approximate formula [10]	present results $P = 0{\cdot}1$	$P = 5{\cdot}0$
1000	3·39	4·626	5·075	3·423	4·44
100	1·573	2·601	2·692	1·60	2·33
10	0·75	1·463	1·6	0·75	1·23

It can be found from Table 5 that the approximate formula predicts higher values. The trend of the results is in agreement with the expected trend reported by Acrivos [10].

The velocity and temperature profiles have been plotted for opposing flows in Fig. 11. It can be observed that velocity profiles for Newtonian and dilatant fluids are much steeper than for pseudoplastic fluid. The thermal boundary layer for pseudoplastic fluid is much thinner than the corresponding ones for Newtonian and dilatant fluids.

The results of heat transfer and shear stress have been plotted in Fig. 12. It can be found that the phenomenon of zero shear at the surface will occur for a dilatant at a lower value of P, than for Newtonian and pseudoplastic fluids indicating the tremendous influence of non-Newtonian behaviour in the opposing flow. For lower values of P, the friction coefficient and heat-transfer rates continuously decrease with the flow behaviour index. The variation of friction coefficient with P is very abrupt in the case of Newtonian and dilatant fluids whereas for pseudoplastics, it is quite gradual.

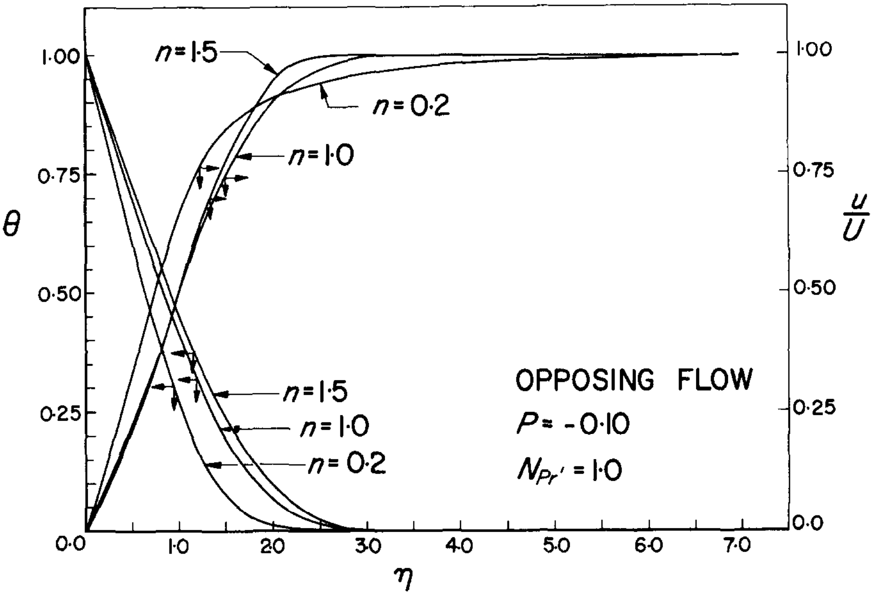

FIG. 11. Representative velocity and temperature profiles.

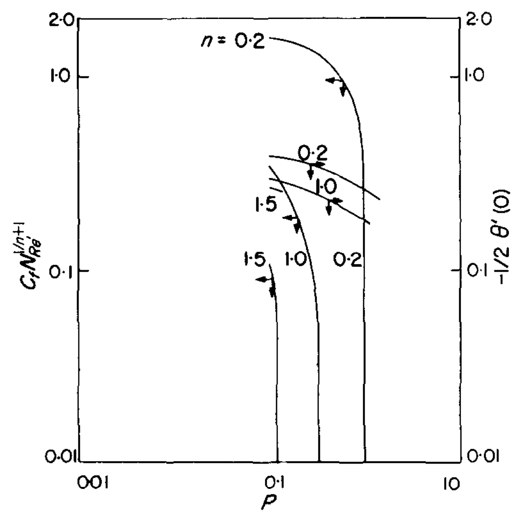

FIG. 12. Plot of shear stress and heat-transfer results, opposing flow.

CONCLUSIONS

1. The combined effect of free and forced convection on non-Newtonian laminar boundary layer can be satisfactorily characterized by the dimensionless group

$$P = N_{Gr'}/N_{Re'}^{2/(2-n)}.$$

2. Based on the two asymptotic formulae in pure flows, it can be found that the effect of free convection is negligible for $P < 0{\cdot}1$ and the effect of forced convection is apparent even at 5·0 for Newtonian fluids. At higher Prandtl numbers the effect of free convection may be predominant at somewhat lower values of P both for Newtonian and non-Newtonian fluids.
3. For opposing flows, the phenomenon of zero shear is strongly influenced by non-Newtonian behaviour.

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Résumé—On présente une analyse théorique pour le transport de chaleur par convection laminaire avec combinaison de convection naturelle et forcée avec des fluides non-newtoniens. Le paramètre gouvernant la convection naturelle et forcée est $N_{Gr'}/N_{Re'}^{2/2-n}$, où n est l'indice de comportement de l'écoulement. La résolution numérique des équations transformées de la couche limite a été effectuée dans le cas d'une plaque plane avec une température pariétale uniforme, pour différents indices de comportement de l'écoulement et différentes valeurs du paramètre qui gouverne le phénomène. Les résultats du transport de chaleur, de la contrainte de cisaillement, des profils de vitesse et de température sont discutés en fonction de l'indice de comportement de l'écoulement pour les écoulements en opposition ou dans le même sens que la convection naturelle.

Zusammenfassung—Es wird eine theoretische Analyse gegeben für den Wärmeübergang an Nicht-Newtonsche Flüssigkeiten in kombinierter freier und erzwungener Aussenströmung. Als anzeigender Parameter für die freie und erzwungene Konvektion ergab sich $N_{Gr'}/N_{Re'}^{2/2-n}$ mit n als Index für das Strömungsverhalten. Die Lösung der transformierten Grenzschichtgleichungen wurde numerisch für die ebene Platte mit gleichförmiger Wandtemperatur durchgeführt mit verschiedenen Indices für das Strömungsverhalten und für unterschiedliche Werte der kennzeichnenden Parameter. Die Ergebnisse des Wärmeübergangs, der Schubspannung, der Geschwindigkeitsund Temperaturprofile als Funktionen des Index des Strömungsverhaltens und der kennzeichnenden Parameter bei verschiedenen Prandtl-Zahlen, werden für gegenläufige und gleichlaufende Strömung diskutiert.

Аннотация—Представлен теоретический анализ переноса тепла при совместной свободной и вынужденной ламинарной конвекции в неньютоновских жидкостях для внешней задачи. Установлено, что свободная и вынужденная конвекция зависят от характерлстического параметра $N_{Gr'}/N_{Re'}^{2/2-n}$, где n — характеристика течения. Проведен численный расчет преобразованных уравнений пограничного слоя для случая плоской изотермической пластины для различных характеристик течения и различных значений характеристического параметра. Полученные результаты по зависимости теплообмена, напряжений сдвига полей скорости и температуры от характеристики течения и характеристического параметра обсуждаются для различных чисел Прандтля для случаев, когда свободная и вынужденная конвекция имеют одно направление и противоположные направления.